ON ADAPTIVE STRATEGY FOR OVERCOME STAGNATION IN LGMRES(m, l)

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Abstract. *The resolution of sparse linear systems is the most time-consuming step in running simulations. The iterative resolution is typically based on projection onto Krylov subspaces. The most popular choice used when linear systems are sparse and non-symmetric is the Generalized Minimal Residual algorithm (GMRES). This method and its variants find the approximate solution in the Krylov subspace minimizing the* 2*-norm of the residual. In particular, we will focus on the method LGMRES* (m, l)*, uses the last* l *error approximations in addition to a Krylov subspace of dimension* m*. These parameters remain constant in each step. This methodology allows to improve the convergence, but can stall (stagnate) if the parameters are not selected correctly. The present work proposes an efficient way to exploit the Krylov subspace information to modify the restart parameters adaptively in order to avoid the stagnation. Numerical results on problems of reservoir simulation show improvements over traditional implementations.*

Keywords: *Iterative method, Krylov subspace, Adaptive GMRES*(m)*, Reservoir simulation.*

1 INTRODUCTION

The Generalized Minimal Residual Method, popularly known as GMRES is a Krylov subspace method for solving large sparse non-symmetric linear systems (Saad and Schultz, 1986)

$$Ax = b \tag{1}$$

where $A \in \mathbb{C}^{n\times n}$ is nonsingular and $b, x \in \mathbb{C}^n$. Let x^0 be an initial approximation and $r^0 = b - Ax^0$ the initial residual. At k-th iterarion, GMRES gives an approximate solution x^k that minimizes the 2-norm of the residual vector $r^k = b - Ax^k$ over all vectors in the k-th Krylov subspace spanned by $\{r^0, Ar^0, \cdots, A^{k-1}r^0\}$. When the iteration proceeds GMRES obtain a sequence of non-increasing residual norms.

In theory, GMRES converges before n iterations, but the cost for computing the orthonormal basis of Krylov subspace increases as the iteration proceeds. Hence GMRES is not practical for large linear systems of equations. To reduce the cost of GMRES, a restarted version of GMRES, which restarts it after each cycle of m iterations, is often used. The restarted version is denoted by GMRES(m). Compared to the GMRES, the restarted version requires less work for storage, but for being competitive in computational cost, an appropriate restart cycle m has to be chosen. Generally, m is selected according by practitioners via some heuristical approach. The best way to select m has not yet been established.

We study a modification of standard GMRES(m), called LGMRES(m, l). This method accelerates the convergence of GMRES(m) augmenting the standard search space using approximations to the errors of previous restart cycles (see Baker et al., 2005). Some interesting properties of the convergence of GMRES(m) motivate the LGMRES(m, l) algorithm developed. Numerical experiments demonstrate that the LGMRES(m, l) augmentation scheme is an effective accelerator for GMRES(m). We propose a new adaptive restarting strategy for the LGMRES(m, l) to speedup standard GMRES(m) and overcome stagnation.

This paper is organized as follows. In §2, we introduce the formulation for LGMRES(m, l), as well as, definitions and ways to identify and characterize the stagnation. In §3, it is presented strategies for overcoming stagnation while numerical results are presented at §4. The conclusion are presented at §5 showing that the adaptive strategy improves the convergence of standard GMRES(m)and LGMRES(m, l).

2 THE ADAPTIVE LGMRES(m, l)

LGMRES(m, l) method is used to accelerate the convergence of the standard GMRES(m), but if the selection of the parameters is not appropriate, the method stagnates. There are several alternatives to avoid the stagnation. For instance, we can mention enriching the subspace of searching through approximations of the eigenvectors (see Morgan, 1995; Chapman and Saad.1997); or modifying the restart parameter m (see Baker et al., 2009; Cuevas and Schaerer, 2010; Cabral and Schaerer, 2014). In this work we focus on the latter strategy, but focusing in the LGMRES(m, l) method. Initially, it is formalized a structural condition to identify when the convergence is deteriorated. For solving part of this problem we propose to use a proportional controller to vary the size of the search subspace for enlarging the search subspace and the parameter m, since decreasing the restart parameter does not contribute to an improvement in the convergence (see Proposition 3.1).

2.1 GMRES(m)

GMRES(m) approximates the solution to system (1) at the j-th restart cycle using the residual at $(j-1)$-th cycle r^{j-1} for constructing a Krylov subspace of $\mathcal{K}_m^j(A, r^{j-1})$ of dimension m. The j-th approximation then is built as

$$x^j \in x^{j-1} + \mathcal{K}_m^j(A, r^{j-1}) \tag{2}$$

where the index m denotes that the restarting parameter was set to the value m, usually constant. GMRES(m) obtain an approximate solution which minimize the 2-norm of the residual $r^j := b - Ax^j$, i.e.,

$$\min_{x^j \in x^{j-1} + \mathcal{K}_m(A, r^{j-1})} \| b - Ax^j \|_2 \ . \tag{3}$$

To solve this problem, the Arnoldi process is normally used for obtaining an orthonormal basis for the Krylov subspace. The first m steps of this procedure can be expressed as:

$$AV_m^j \ = \ V_{m+1}^j \tilde{H}_m^j \tag{4}$$

where $V_m^j \in \mathbb{C}^{n \times m}$ and $V_{m+1}^j := [V_m^j \, v_{m+1}^j] \in \mathbb{C}^{n \times (m+1)}$ have orthonormal columns and $\tilde{H}_m^j \in \mathbb{C}^{(m+1) \times m}$ is the upper Hessenberg matrix formed by an upper matrix H_m^j of dimension $m \times m$ and an entry $h_{m+1,m}^j$ placed at position $(m+1, m)$. If the Arnoldi process starts with $v_1^j = (\frac{1}{\beta}) r^{j-1}$, where $\beta = \| r^{j-1} \|_2$, then by construction the columns of V_m^j are an orthogonal basis of the subspace $\mathcal{K}_m(A, r^{j-1})$. Defining at j-th cycle of GMRES(m), the functional $J(y^j)$, then

$$J(y^j) = \| b - Ax^j \|_2 = \| b - A(x^{j-1} + V_m^j y^j) \|_2 \tag{5}$$

and using the Arnoldi relation

$$\begin{aligned} J(y^j) &= \| b - Ax^{j-1} - AV_m^j y^j \|_2 \\ &= \| r^{j-1} - AV_m^j y^j \|_2 = \| \beta v_1^j - V_{m+1}^j \tilde{H}_m^j y^j \|_2 \\ &= \| V_{m+1}^j \beta e_1^{(m+1)} - V_{m+1}^j \tilde{H}_m^j y^j \|_2 \\ &= \| V_{m+1}^j (\beta e_1^{(m+1)} - \tilde{H}_m^j y^j) \|_2, \end{aligned}$$

where $e_i^{(n)}$ is the i-th column of the $n \times n$ identity matrix. As a consequence,

$$J(y^j) = \| b - Ax^j \|_2 = \| \beta e_1^{(m+1)} - \tilde{H}_m^j y^j \|_2 \tag{6}$$

The GMRES(m) approximation gives a vector y^j which minimizes (6), and the minimization of functional (6) with respect to y^j is equivalent to the minimization of the expression (3). The approximate solution x^j is then obtained by $x^j = x^{j-1} + V_m^j y^j$ (Saad, 2003). For solving expression (6) by least squares, in practice it is computed a QR decomposition of matrix $\tilde{H}_m^j$ using plane Givens rotations (see Saad, 2003; Eiermann et al., 2000).

We consider now the angles between the residual at cycles. The angles $\alpha^j := \angle(r^j, r^{j-1})$ and $\gamma^j := \angle(r^j, r^{j-2})$ are named sequential and skip angles, respectively. In accordance with Theorem 4 in (Baker et al., 2005), the sequential angle of GMRES(m) can be defined implicitly by

$$\cos(\alpha^j) := \frac{\| r^j \|_2}{\| r^{j-1} \|_2}$$

When the residual vectors point in nearly the same direction at the end of every restart cycle, the skip angle is small. This allows to characterize the slow convergence, i.e., $\angle(r^j, r^{j-1}) \approx 0$, because $\| r^j \|_2 \approx \| r^{j-1} \|_2$.

2.2 LGMRES

This algorithm proposed in (Baker et al., 2005) uses the simple framework of Morgan's GMRES-E(m, d) method (Morgan, 1995) for appending vectors to the standard Krylov space. The motivation of LGMRES(m, l) is based on preventing an alternating behavior observed in the GMRES(m) residual at consecutive cycles which results in deteriorating the convergence.

To prevent the alternating behavior, LGMRES include approximations to the error in the current approximation space. We define the approximation to the error after the j-th restart cycle as

$$\varphi^j = x^{j-1} - x^{j-2} \tag{7}$$

and $\varphi^j \equiv 0$ for $j < 1$. This error approximation vector is used for augmenting the approximation space $\mathcal{K}_m(A, r^{j-1})$ at the next cycle. Note that $\varphi^j \in \mathcal{K}_m(A, r^{j-1})$. Therefore, this error approximation φ^j in some sense represents the space $\mathcal{K}_m(A, r^{j-1})$ generated in the previous cycle and subsequently discarded. Hence it is a natural choice for enriching the next approximation space $\mathcal{K}_m(A, r^j)$.

The method LGMRES(m, l) augmentes the standard Krylov approximation space with l previus approximations to the error. Therefore, at the end of restart cycle j, LGMRES(m, l) finds an approximate solution to linear system in the following way:

$$x^j = x^{j-1} + q^j_{m-1} + \sum_{k=(j-l+1)}^{j} \alpha_{jk}\varphi^k \tag{8}$$

where polynomial q^j_{m-1} and α_{jk} are chosen such that $\| r^j \|_2$ is minimized. Note that $l = 0$ corresponds to standard GMRES(m).

The augmented approximation space $S = \mathcal{K}_m(A, r^{j-1}) \cup span\{\varphi^k\}, k = (j-l+1) : j$ has dimension $s \equiv m + l$. We find the approximate solution from S whose corresponding residual is a minimum in the 2-norm. The matrix V_{s+1} is the $n \times (s + 1)$ orthonormal matrix whose first $m + 1$ columns are the Arnoldi vectors and last l columns result from orthogonalizing the l error approximation vectors ($\varphi^k, k = (j - l + 1) : j$) against the previous columns of Arnoldi vectors. W^j_s is the $n \times s$ matrix whose first m columns are equal to the first m columns of V^j_{s+1} and whose last l columns of W^j_s are the l error approximation vectors (typically normalized so that all columns are of unit length). Then the relationship

$$AW^j_s = V^j_{s+1}\tilde{H}^j_s \tag{9}$$

holds for LGMRES(m, l), $\tilde{H}^j_s$ denotes an $(s + 1) \times s$ upper Hessenberg matrix. Typically, the number l of vectors appended is much smaller than the restart parameter m (Baker et al., 2005).

It is important to remark that LGMRES is not helpful when one of the following situations occurs: (a) when GMRES(m) skip angles are not small; (b) when GMRES(m) sequential angles vary greatly from cycle to cycle; (c) when GMRES(m) converges in a small number of iterations; or (e) when GMRES(m) skip angles and sequential angles are near zero (indicating stagnation). LGMRES is not typically a substitute for preconditioning and does not help when a problem stalls for a given restart parameter. Possible improvements to the algorithm include a robust adaptive variant (Baker et al., 2005).

3 Adaptive-LGMRES(m, l)

The method combines two existing techniques in order to avoid slow convergence and stagnation. We modify a method proposed by Cabral and Schaerer (2014). Firstly we study the possibility of augmenting the dimension of the Krylov subspace increasing the m parameter and using the approximation of errors instead of approximations of eigenvectors.

We observe three possibilities in the m value for the next cycle: decreasing, remaining constant and increasing. From the point of view of present proposal the first two possibilities do not avoid stagnation. Observe this in the following proposition.

Proposition 3.1. During a stagnation at the GMRES(m), we can obtain a decrease in the residual norm at the next restart cycle if we increase the value of parameter m.

Proof. According to Lemma 2.5 of Tebbens and Meurant, 2015 and Theorem 3.1 of Strikwerda and Stodder, 1995, when GMRES(m) presents a stagnation, the first row of the Hessenberg matrix has zeros components.

$$J(y^j) = \left\| \begin{pmatrix} \beta \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \cdots & 0 \\ h^j_{2,1} & h^j_{2,2} & \cdots & h^j_{2,m} \\ 0 & h^j_{3,2} & \cdots & h^j_{3,m} \\ \vdots & \ddots & \vdots & \\ 0 & \cdots & h^j_{m,m-1} & h^j_{m,m} \\ 0 & \cdots & 0 & h^j_{m+1,m} \end{pmatrix} \begin{pmatrix} y^j_1 \\ y^j_2 \\ \vdots \\ y^j_m \end{pmatrix} \right\|_2$$

Because y^j minimize $J(y^j)$ and the matrix without the first row is upper triangular with linearly independent rows, the residual norm at j-th cycle is $\| r^j \|_2 = \| r^{j-1} \|_2$

Firstly, we consider the decrease in one unit at the j-th restart cycle,

$$J(\tilde{y}^j) = \left\| \begin{pmatrix} \beta \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \cdots & 0 \\ h^j_{2,1} & h^j_{2,2} & \cdots & h^j_{2,m-1} \\ 0 & h^j_{3,2} & \cdots & h^j_{3,m-1} \\ \vdots & \ddots & \vdots & \\ 0 & \cdots & h^j_{m-1,m-2} & h^j_{m-1,m-1} \\ 0 & \cdots & 0 & h^j_{m,m-1} \end{pmatrix} \begin{pmatrix} \tilde{y}^j_1 \\ \tilde{y}^j_2 \\ \vdots \\ \tilde{y}^j_{m-1} \end{pmatrix} \right\|_2$$

Thus, there is no improvement and the vector $\tilde{y}^j$ is zero and it is established analogously for any decrease in m. In the next place, we consider an increase of m in a unit,

$$J(\hat{y}^j) = \left\| \begin{pmatrix} \beta \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \cdots & h^j_{1,m+1} \\ h^j_{2,1} & h^j_{2,2} & \cdots & h^j_{2,m+1} \\ 0 & h^j_{3,2} & \cdots & h^j_{3,m+1} \\ \vdots & \ddots & \vdots & \\ 0 & \cdots & h^j_{m,m-1} & h^j_{m,m+1} \\ 0 & \cdots & 0 & h^j_{m+2,m+1} \end{pmatrix} \begin{pmatrix} \hat{y}^j_1 \\ \hat{y}^j_2 \\ \vdots \\ \hat{y}^j_{m+1} \end{pmatrix} \right\|_2$$

Considering a fixed value of m we have $\| r^{j-1} \|_2 = \beta$, the first element of the vector $J(y^j)_1 = \beta$. But for the case of an increase in a unit we will have: $J(y^j)_1 = \beta - (0 \times \hat{y}_1 + \cdots + 0 \times \hat{y}_m + h_{1,m+1} \times \hat{y}_{m+1})$. This allows, $\| r^j \|_2 < \| r^{j-1} \|_2$, as long as the $h_{1,m+i} \times \hat{y}_{m+i} \neq 0, \forall i = 1 : \alpha_P$. If we consider an increase greater than one, we would have:

$$J(\hat{y}^j) = \left\| \begin{pmatrix} \beta \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \cdots & h^j_{1,m+1} & \cdots & h^j_{1,m+\alpha_P} \\ h^j_{2,1} & h^j_{2,2} & \cdots & h^j_{2,m+1} & \cdots & h^j_{2,m+\alpha_P} \\ 0 & h^j_{3,2} & \cdots & h^j_{3,m+1} & \cdots & h^j_{3,m+\alpha_P} \\ \vdots & \ddots & \vdots & & & \\ 0 & \cdots & h^j_{m,m-1} & h^j_{m,m+1} & \cdots & h^j_{m,m+\alpha_P} \\ 0 & \cdots & 0 & h^j_{m+\alpha_P,m+1} & \cdots & h^j_{m+\alpha_P+1,m+\alpha_P} \end{pmatrix} \begin{pmatrix} \hat{y}^j_1 \\ \hat{y}^j_2 \\ \vdots \\ \hat{y}^j_{m+1} \\ \vdots \\ \hat{y}^j_{m+\alpha_P} \end{pmatrix} \right\|_2$$

which will allow, $J(\hat{y}^j) = \beta - (0 \times \hat{y}^j_1 + \cdots + 0 \times \hat{y}^j_m + h^j_{1,m+1} \times \hat{y}^j_{m+1} + \cdots + h^j_{1,m+\alpha_P} \times \hat{y}_{m+\alpha_P})$, thus $\| r^j \|_2 < \| r^j \|_2$. □

Proposition 3.2. If we have stagnation, appending error approximation vectors to the standard Krylov space can not remove the stagnation.

Proof. If we have complete stagnation, imply $\varphi^j = x^{j-i} - x^{j-i-1} = 0$. In this way we have,

$$(v^j_1, A\varphi^j) = 0 \Rightarrow r^j = r^{j-1}$$

□

Our new method is denoted as A-LGMRES(m, l), where at the same time to increase the search subspace with information of the previous cycles, it is increasing the value of m when the algorithm presents stagnation. We enrich the Krylov subspace with l error approximation. In this way, the search subspace will be of size $s = m_j + l$. W_s is a $n \times s$ matrix with the first m_j columns being the Arnoldi's vectors and the last l vectors being error approximation $\varphi^i, \forall i = 1, \cdots, l$. This strategy improves the convergence when the method converges and has no stagnation problems due to bad selection of the restart parameter.

At the end of jth restart cycle, A-LGMRES(m, l) seeks the approximate solution x^j of the form

$$x^j = x^{j-1} + z^j \qquad \text{with} \qquad z_j \in \mathcal{K}_m(A, r^{j-1}) + \{\varphi^i\}^l_{i=1} \tag{10}$$

where $\mathcal{K}_m(A, r^{j-1}) + \{\varphi^i\}_{i=1}^{l}$ is the enriched search subspace.

In addition, to the enriched search subspace, an adaptive way to vary the restart parameter is introduced. This is done for avoiding a possible stagnation for any bad selection of the restart parameter. We use an updating law for m_j named discrete Proportional controller with the form:

$$m_j = m_{j-1} + \alpha_P \left\lfloor \frac{\|r^{j-1}\|}{\|r^{j-2}\|} \right\rceil. \tag{11}$$

where $\alpha_P \in \mathbb{Z}$ and $\left\lfloor \frac{\|r^{j-1}\|}{\|r^{j-2}\|} \right\rceil$ means the nearest integer to $\frac{\|r^{j-1}\|}{\|r^{j-2}\|}$. Observe that poor convergence implies $\frac{\|r^{j-1}\|}{\|r^{j-2}\|} \approx 1$. According the Proposition 3.1, the selection of $\alpha_P > 0$ is taken for increasing the restart parameter in the next cycle. On the other hand, if the algorithm has good convergence $\frac{\|r^{j-1}\|}{\|r^{j-2}\|} \approx 0$, and the restart parameter remain constant ($m_j = m_{j-1}$).

Baker et al., (2005) has demostrated that the optimal number l of error approximations used in LGMRES at every cycle is typically small, generally less or equal than 3. We observe in this paper that the optimal value for l is also small (see Section 4.1). The pseudocode for the j-th cycle of the proposede method called A-LGMRES(m, l) is presented in the Algorithm 1.

Algorithm 1 The j-th cycle of A-LGMRES(m,l), ($j > l$).

Require: Given A, x^{j-1},r^{j-1},m_j,φ^j.

$s = m_j + l$
for $k = 1 : s$ **do**

$$w = \begin{cases} Av_k, & j \leq m_j \\ \\ A\varphi_{k-m_j}, & m_j + l + 1 \leq k \leq m_j + l \end{cases}$$

  for $i = 1 : k$ **do**
    $h_{i,k} =< w, v_i >$
    $w = w - h_{i,k} v_i$
  end for
  $h_{k+1,k} =\| w \|_2$
  if $h_{k+1,k} = 0$ **then**
    stop;
  end if
  $v_{k+1} = w/h_{k+1,k}$
end for
Find $y^j = argmin_{y \in \mathbb{C}^s} \| \beta e_1 - \tilde{H}_s y \|_2$, compute x^j and r^j;
if $\| r^j \|_2 < tolerance$ **then**
  stop;
else
  Compute the error approximations vectors, φ^j, $j = j - l + 1, \ldots, j$:
  Compute m_{j+1} from Proportional Derivative strategy;
end if
$j = j + 1$

4 NUMERICAL RESULTS

We show the potential of the proposed method by presenting experimental results from a variety of problems. We tested 12 problems from the Matrix Market Collection (all problems of fluid mechanics). A zero initial guess is used for all problems. If a right-hand side is not provided, we generate a random right-hand side. We stop the algorithm when the relative residual norm is less than the convergence tolerance, i.e., when $\frac{\|r^j\|_2}{\|r^0\|_2} \leq 10^{-9}$ or when the maximum number of cycles is exceeded ($j \leq 1000$).

The matrices used in this work are non-symmetric stored in Compressed Sparse Column format (CSC). These matrices are presented in Table 1 . The columns labeled $size$ and nnz are for matrix dimension and number of non-zeros entries. $Condest$ refers to the condition number. All algorithms were implemented in MATLAB. The implementations are own using pseudocodes proposed in the article of Saad and Schultz (1986), so that we can make changes to the restart parameters within cycles and enrich the search subspace. The tests are run on a desktop machine with Intel Core i3-2310M CPU @ 2.10GHz X 4 and 5.8 GB of main memory, by using MATLAB 7.7.0 for Linux.

In the subsection §4.1 we select the parameters for the proposed method. The comparison between the methods GMRES(m), LGMRES(m, l), and A-LGMRES(m, l) is given in §4.2. Then, in subsection §4.3 is discussed the A-LGMRES(m, l) for avoiding the stagnation, while GMRES(m) and LGMRES(m, l) fail.

Table 1: The matrices information.

Matrix	size	nnz	Condest
raefsky1	3242	294276	3.16E+4
raefsky2	2242	293551	1.08E+4
sherman1	1000	3750	2.26E+4
sherman4	1104	3786	7.16E+3
orsreg1	2205	14133	1.54E+3
orsirr1	1030	6858	1.67E+5
rdb2048	2048	12032	2.09E+2
steam2	600	13760	3.55E+3
cavity05	1182	32632	9.18E+5
sherman5	3312	20793	3.90E+5
sherman3	5005	20033	6.90E+16
cavity10	2597	76171	4.46E+6

4.1 Selection of parameters to A-LGMRES(m, l)

We chose the initial restart parameter $m = 30$ because it is a common choice and often the default in general linear solver packages such as PETSc (see Balay et al., 2001). For GMRES

(m) the restart parameter remain constant. The total size of search subspace of LGMRES(m, l) considered in this paper is 30, i.e., we will use LGMRES($30 - l, l$) to compare with the other methods. Similarly for the A-LGMRES method, the size of the search subspace for the start is the same as in the previous case, but in this case the value of m can increase when there are drawbacks in the convergence. For the last two cases, when we have $j < l$, there are no enough error vectors, so it is recommended using Arnoldi vectors instead of φ^j such that the dimension of the search subspace can be fixed as $m + l$.

Firstly, the parameter α_P is selected for allowing an increasing in the value of m according (see §3). The parameter α_P must be positive to increase m_j using the proportional controller proposed in §3. If an stagnation is observed, high values of α_P increases the computational cost and storage requirements, hence the good property of GMRES(m) is lost. In contrast, small values allow the possibility of the residual norm decrease, but can remain stagnated in the next restart.

For the set of problems we have tested values of α_P between 0 and 5 (see Table 2). For $\alpha_P > 5$, a lower cycle is achieved but with a longer time, this is due to the fact that the cost of orthogonalization increases in problems with slow convergence or stagnation. For this test, we use $l = 3$, which is within the range of optimal values proposed by Baker, (2005).

Table 2: The performance of A-LGMRES(m, l) for different α_P and considering number of restart cycle.

Matrix	α_P: proportional constant to increase m					
	0	1	2	3	4	5
raefsky1	29	24	22	21	19	18
raefsky2	59	31	27	25	24	21
sherman1	28	24	21	20	20	18
sherman4	14	13	12	12	12	12
orsreg1	20	18	16	15	15	15
orsirr1	75	37	36	30	28	26
rdb2048	17	16	16	14	14	14
steam2	20	12	11	11	9	9
cavity05	81	32	29	26	24	23
sherman5	1000	111	73	66	53	52
sherman3	289	77	62	52	46	43
cavity10	130	44	39	35	32	29

Table 3 shows the summatory of the cycles required for convergence for each α_P value and its corresponding time consuming for the 12 problems tested. If we use $\alpha_P = 0$, we have the standard LGMRES(m, l). In this work, we select the value $\alpha_P = 2$ since it presents the lowest value in the summatory of cycles in Table 3. The other option would be α_P =

1, however this represents a 6.7% of the increase over the smallest summatory of time. The standard LGMRES(m, l) has an increase of 30% with respect to the proposed method with the selected parameter.

Table 3: Sum of cycles and time for every α_P.

Matrix	α_P: proportional constant to increase m					
	0	1	2	3	4	5
Sum of cycles	1762	439	364	327	296	280
Sum of time (seconds)	49.38	39.94	37.44	42.90	42.05	50.38

Next, we select the parameter l. The optimal values for l are typically very small, generally $l \leq 3$ (see Baker, 2005). For this work, we consider $1 \leq l \leq 5$ and search the best value of l to get the shortest time and the least number of cycles.

The number of restarts cycle and the time (in seconds) to make the relative residual norm below 10^{-9} are shown for different values of l in Table 4 and Table 5, respectively. We can see that A-LGMRES(m, l) with $l = 3$ and $l = 5$ produces better results on most of the tested matrices. For this work, we select $l = 3$ being the maximum optimal value proposed by Baker, (2005); and being equal to the initial consideration for α_P. The proposed method using $l = 3$ has an increase of only 4% with respect to the smallest sumatory of time obtained with $l = 5$.

Table 4: The performance of A-LGMRES(m, l) for different l, considering number of restart cycle.

Matrix	l: number of error approximations				
	1	2	3	4	5
raefsky1	27	22	22	19	18
raefsky2	36	27	27	25	24
sherman1	24	22	21	21	21
sherman4	16	14	12	11	11
orsreg 1	16	16	16	16	16
orsirr 1	39	37	36	35	34
rdb2048	16	15	16	14	15
steam2	14	13	11	11	8
cavity05	36	33	29	28	28
sherman5	87	74	73	78	70
sherman3	70	61	62	61	60
cavity10	45	44	39	38	32

Table 5: The performance of A-LGMRES(m, l) for different l, considering time in seconds.

Matrix	l: number of error approximations				
	1	2	3	4	5
raefsky1	1.94	1.36	2.29	1.15	1.09
raefsky2	2.97	1.88	2.07	1.85	1.73
sherman1	0.49	0.39	0.38	0.43	0.39
sherman4	0.18	0.12	0.11	0.11	0.12
orsreg 1	0.3	0.29	0.29	0.3	0.36
orsirr 1	0.8	0.76	0.72	0.79	0.8
rdb2048	0.3	0.25	0.3	0.25	0.29
steam2	0.14	0.1	0.08	0.08	0.06
cavity05	1.84	1.71	1.23	1.36	1.22
sherman5	23.39	16.51	17.11	20.69	16.52
sherman3	17.88	12.58	12.75	13.48	14.07
cavity10	3.25	3.38	2.95	2.69	2.04

4.2 Comparison to standard methods

In this section, numerical examples are given to demonstrate the improvement of LGMRES(m, l) method, obtained from the adaptive variation of the restart parameter using $\alpha_P = 2$. The numerical results are presentented in Table 6.

Table 6: Numerical results for orsreg1, orsirr1, steam2 and cavity05.

Matrix	GMRES(30)		LGMRES(27, 3)		A-LGMRES(27, 3)	
	cycle	time	cycle	time	cycle	time
orsreg1	26	0.38	20	0.27	16	0.27
orsirr1	121	0.9	76	0.59	36	0.76
steam2	12	0.07	20	0.12	11	0.07
cavity05	825	10.04	81	2.36	29	1.59

Example 1. We consider the matrices orsirr 1 and cavity05. Both matrices are real non-symmetric from computational fluid problems. No problems of stagnation were found using the standard algorithms. The proposed method have the smallest number of cycles but a little increase in the time of convergence for the case of orsirr1 comparing with the LGMRES($27, 3$) (see Figure 1). In the case of cavity05, the proposed method have the smallest number of cycles and time of convergence(see Figure 2).

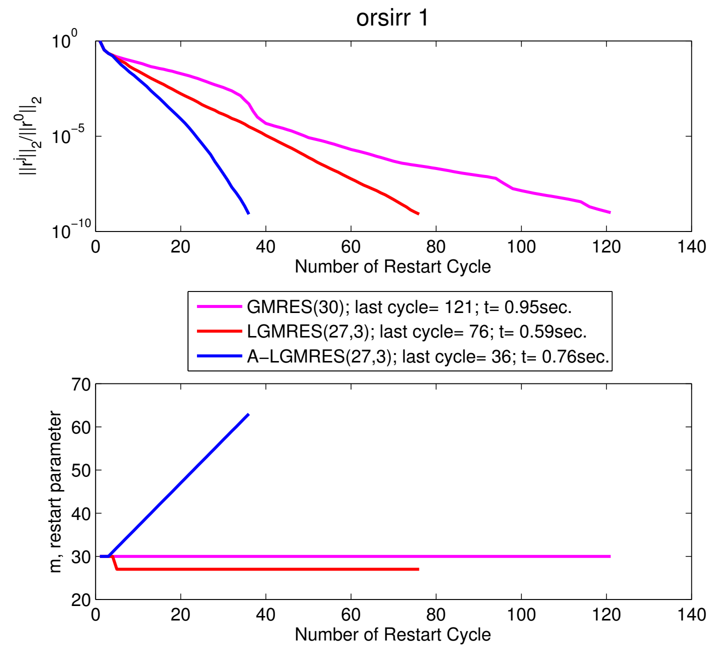

Figure 1: The convergence curves of matrix orsirr1.

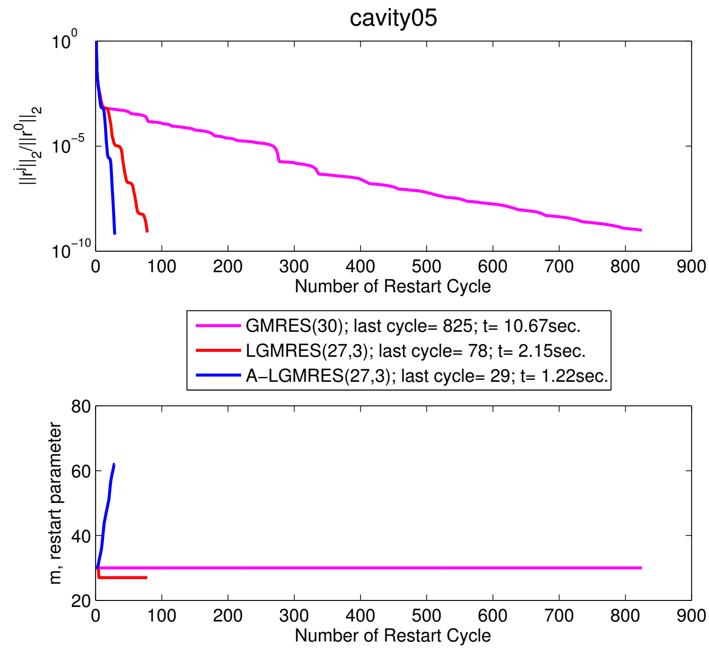

Figure 2: The convergence curves of matrix cavity05.

4.3 Overcoming the stagnation

LGMRES(m, l) acts an accelerator for GMRES(m) but is not designed to overcome stalling as the error approximation vectors, φ^j, are zero or close to zero when the residual norm does not decrease within a cycle (see Proposition 3.2). In this work, three matrices: sherman3, sherman5 and cavity10, have stagnation using GMRES(30). We compare the proposed method with

LGMRES(m, l) and GMRES-E(m, d) (see Morgan, 1995). The numerical results are presentented in Table 7. If the number of cycle is 1000, it implies that the tolerance established in the stop criterion was not achieved and the method failed.

Table 7: Numerical results for sherman5, sherman3 and cavity10.

Matrix	LGMRES$(27, 3)$		GMRES-E$(27, 3)$		A-LGMRES$(27, 3)$	
	cycle	time	cycle	time	cycle	time
sherman5	1000	31.85	266	11.28	73	16.34
sherman3	311	10.67	1000	52.68	62	13.15
cavity10	113	4.04	188	10.16	39	2.80

Example 2. Simple example showing stagnation problem in the convergence of GMRES(30). We consider the matrices sherman3 and sherman5 generated in oil reservoir simulation. Both matrices are real non-symmetric with relative small eigenvalues in magnitude. The convergence results are shown in Figure 3 and Figure 4.

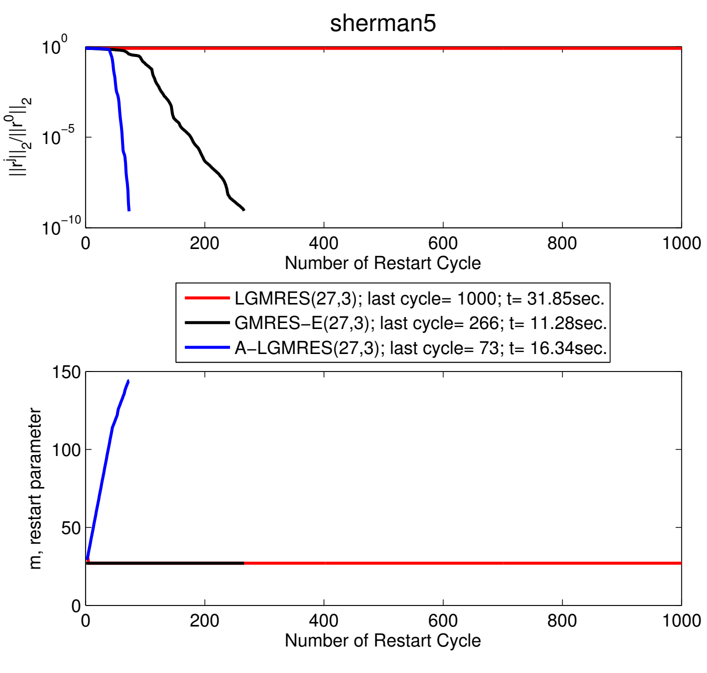

Figure 3: The convergence curves of matrix sherman5.

5 CONCLUSION

An Adaptive LGMRES for solving linear systems has been presented and tested, as well as a criterion to vary the restart parameter. This criterion is based on the presence of stagnation. It is important to remark that the convergence is improved for all tested cases. Moreover,

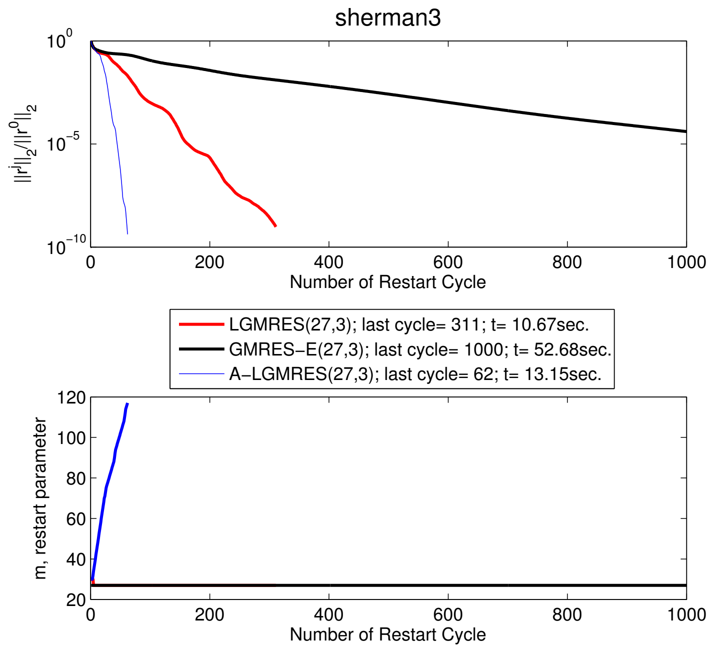

Figure 4: The convergence curves of matrix sherman3.

in order to achieve an improvement in terms of time of convergence, it is important to choose adequately the proportional parameter. The result is quite encouraging since it is possible to obtain improvements in both the convergence and the computational cost with respect to standard methods.

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REFERENCES

Balay S., Buschelman K., Gropp W. D., Kaushik D., Knepley M.G., McInnes L.C., Smith B.F., & Zhang H. PETSc Web page, 2001. https://www.mcs.anl.gov/petsc/.

Baker, A. H., Jessup, E. R., & Manteuffel, T., 2005. A technique for accelerating the convergence of restarted GMRES. SIAM Journal on Matrix Analysis and Applications, 26(4), 962-984.

Baker, A. H., Jessup, E. R., & Kolev, T. V. , 2009. A simple strategy for varying the restart parameter in GMRES (m). Journal of computational and applied mathematics, 230(2), 751-761.

Chapman, A., & Saad, Y., 1997. Deflated and augmented Krylov subspace techniques. Numerical linear algebra with applications, 4(1), 43-66.

Cuevas, R., & Schaerer C., 2010. A control inspired strategy for varying the restart parameter m of GMRES(m). In Congresso Nacional de Matemtica Aplicada e Computacional, pages 1000-1001, CNMAC XXXIII.

Cabral, J.C., & Schaerer C., 2014. Harmonic Ritz control strategy for restarting GMRES(m). In Third Conference of Computational Interdisciplinary Sciences., pages 133-138, CCIS 2014.

Morgan, R. B., 1995. A restarted GMRES method augmented with eigenvectors. SIAM Journal on Matrix Analysis and Applications, 16(4), 1154-1171.

Eiermann, M., Ernst, O. G., & Schneider, O., 2000. Analysis of acceleration strategies for restarted minimal residual methods. Journal of Computational and Applied Mathematics, 123(1), 261-292.

Saad, Y. & Schultz, M. H., 1986. GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems. SIAM Journal on scientific and statistical computing, 7(3), 856-869.

Saad, Y., 2003. Iterative methods for sparse linear systems. SIAM. Philadelphia. USA.

Strikwerda, J. C., & Stodder, S. C. , 1995. Convergence results for GMRES (m). Department of Computer Sciences, University of Wisconsin. USA.

Tebbens,J. D. & Meurant, G. 2015. On the admissible convergence curves for restarted GMRES.